Lectures on Berry phase related phyiscs

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Contents of lectures

Topics

- Berry phase and physical properties
- Berry phase and physical properties 2
- Berry curvature and anomalous Hall conductivity
- Topological invariant: Chern number and Z_2
- Computing Chern number and Z_2 as local quantity

First-order differential equation

Variation of parameters

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

Solution

- Solve *homogeneous equation* $\frac{dy}{dx} + p(x)y = 0$
  $y_0 = C_0 e^{\int p(x)dx}$
  $C_0 \to C(x)$ and substitue $y = C(x)e^{\int p(x)dx}$ into equation (1).
- $C(x) = \int q(x)e^{\int p(x)dx}dx$
- Solution: $y = C_0 e^{\int p(x)dx} + e^{\int p(x)dx} \int q(x)e^{\int p(x)dx}$

Time-dependent Schrödinger equation

$$H(t+T) = H(t)$$

$$i\hbar\frac{\partial\psi}{\partial t} = H(t)\psi$$

$$H(t)\phi_n(t) = E_n(t)\phi_n(t)$$

$$\psi = \sum_n C_n(t)\phi_n(t) \simeq C_n(t)\phi_n(t)$$

$$i\hbar\left(\phi_n\frac{dC_n}{dt} + C_n\frac{d\phi_n}{dt}\right) = H(t)C_n\phi_n = E_nC_n\phi_n$$

Time-dependent Schrödinger equation

Solution by using Variation of parameters method:

$$i\hbar\phi_n\frac{dC_n}{dt} - E_n\phi_n C_n = -i\hbar C_n\frac{d\phi_n}{dt} \tag{2}$$

Homogeneous equation:

$$i\hbar\frac{dC_n}{dt} = E_n C_n$$

$$\log C_n = -\frac{i}{\hbar}\int_0^t E_n(s)ds$$

$$C_n = C_0 \exp\left(-\frac{i}{\hbar}\int_0^t E_n(s)ds\right)$$

$$C_0 \to C_0(t) = C_0 e^{i\theta(t)}$$

Berry phase

Time-dependent Schrödinger equation

$$i\hbar\left(\phi_n \frac{dC_n}{dt} + C_n \frac{d\phi_n}{dt}\right) = E_n C_n \phi_n$$

$$i\hbar\left(e^{i\theta(t)}\phi_n \frac{dC_n}{dt} + C_n e^{i\theta(t)}\phi_n \left(i\frac{d\theta(t)}{dt}\right) + C_n e^{i\theta(t)}\frac{d\phi_n}{dt}\right) = E_n C_n e^{i\theta(t)}\phi_n$$

$$i\hbar\left(\phi_n \frac{dC_n}{dt} + C_n \phi_n \left(i\frac{d\theta(t)}{dt}\right) + C_n \frac{d\phi_n}{dt}\right) = E_n C_n \phi_n$$

$$\phi_n \left(\frac{d\theta(t)}{dt}\right) = i\frac{d\phi_n}{dt}$$

$$\int dr \phi_n^* \phi_n \left(\frac{d\theta(t)}{dt}\right) = \int dr \left(i\phi_n^* \frac{d\phi_n}{dt}\right)$$

Berry phase

Time-dependent Schrödinger equation

$$\int dr \phi_n^* \phi_n \left(\frac{d\theta(t)}{dt}\right) = \int dr \left(i\phi_n^* \frac{d\phi_n}{dt}\right)$$

$$\theta(t') = \int_0^{t'} dt \int dr \left(i\phi_n^* \frac{d\phi_n}{dt}\right)$$

Berry phase

$$\theta(T) - \theta(0) = \frac{1}{\hbar}\int_0^T dt \int dr \phi_n^* \frac{\partial \phi_n}{\partial t} \neq 0$$

$$\frac{\partial}{\partial t} = \frac{dx}{dt}\frac{\partial}{\partial x}$$

$$\theta(x(T)) - \theta(x(0)) = \frac{1}{\hbar}\int_{x(0)}^{x(T)} dx' \int dr \phi_n^* \frac{\partial \phi_n}{\partial x'} \neq 0$$